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## STABLE LAWS AND THE DISTRIBUTION OF CASH-FLOWS

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# Stable laws and the distribution of cash-flows

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**Abstract**

In the present contribution, we consider the present value of a series of cash flows under stochastic interest rates and we make use of stable laws to model these interest rates. Just as in some previous papers, we will not try to calculate the exact analytical distribution for the cash-flow, but instead we determine convex upper bounds with an easier structure, and we derive results for the stop-loss premium and distribution of these bounds.

# 1 Introduction

In some earlier contributions (see [1, 2, 7]) we investigated the present value $A$ of a series of $n$ payments at times $t_1 < \ldots < t_n$:

$$A = \sum_{j=1}^{n} c_j e^{-Y(t_j)}, \tag{1}$$

where $Y(t_j)$ represents the stochastic continuous compounded rate of return over the period $[0, t_j]$.

Up to now, the results based on comonotonic risks have been applied to the case of Wiener processes. In fact, we wrote the rate of return as the sum of

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increments over the previous periods,

$$Y(t_j) = \sum_{i=1}^{j} (Y(t_i) - Y(t_{i-1})) \tag{2}$$

where $0 = t_0 < t_1 < \ldots < t_n = t$, and where each jump $Y(t_i) - Y(t_{i-1})$ denotes the rate of return for the period $[t_{i-1}, t_i]$.

Since the stochastic process $\{Y(s)\}_{s \geq 0}$ is assumed to be a Wiener process, the increments $Y(t_i) - Y(t_{i-1})$ are independent and normally distributed:

$$Y(t_i) - Y(t_{i-1}) \stackrel{d}{=} \mu(t_i - t_{i-1}) + (t_i - t_{i-1})^{1/2} \sigma Z_i \tag{3}$$

with $Z_i \sim N(0, 1)$ independent standard normal variables. As a consequence of the properties of Wiener processes, for the compounded rates of return we have

$$Y(t_j) \stackrel{d}{=} \mu t_j + t_j^{1/2} \sigma X_j \tag{4}$$

with $X_j \sim N(0, 1)$ a standard normal variable. Note that $Y(t_j) = Y(t_{j-1}) + (Y(t_j) - Y(t_{j-1}))$, so the variables $Y(t_j)$ and $Y(t_{j-1})$ are not independent.

Financial data, such as logreturns, often turn out to be heavy-tailed and (slightly) skewed, so a Wiener process might not be appropriate. The other limiting distributions of the Generalized Central Limit Theorem, i.e. the stable distributions, do allow for heavy tails and skewness. In a paper of 1994, Kozubowski and Rachev investigated the performance of stable laws and in particular geometric stable laws when modeling asset returns. One of their most important findings was that these models provide a good fit for financial data sets. Hence, in the present contribution we will model the increments by means of a stable distribution and at the same time, we will introduce a risk parameter $\Theta$. For a particular choice of the distribution of $\Theta$ the geometric stable distribution arises.

The aim of this paper is the investigation of the random present value of (1) when rates of return are modelled by the generalized model. The paper is organized as follows. First we will give a summary of the concepts, properties and methods that are needed to reach our goal. In section 2 we briefly describe the stable laws and use them to construct the generalized model for the increments. Section 3 provides the methodology of bounds in convexity order. In section 4, we will be able to present the results about the present value in (1). Finally section 5 gives numerical illustrations of the results of section 4.

## 2 The generalized model

The normal distribution is commonly used in financial data modelling. Perhaps the most famous application is the Black-Scholes model for asset logreturns. A nice feature of the normal distribution is its *stability* property.

**Definition 2.1.** *A random variable $X$ is stable (in the broad sense) if for $X_1$ and $X_2$ independent copies of $X$ and any positive constants $a$ and $b$,*

$$aX_1 + bX_2 \stackrel{d}{=} cX + d, \tag{5}$$

*for some positive $c$ and some $d \in \mathbb{R}$. The random variable is strictly stable (or stable in the narrow sense) if (5) holds with $d = 0$ for all choices for $a$ and $b$.*

From the Generalized Central Limit Theorem, see [5], we know that the stable distributions are the only possible non-trivial limit of normalized sums of independent and identically distributed terms.

**Theorem 2.2 (Generalized Central Limit Theorem).** *Let $X_1, X_2, \ldots$ be a series of independent and identically distributed random variables. There exist constants $a_n > 0$, $b \in \mathbb{R}$ and a non-degenerate random variable $Z$ with*

$$a_n(X_1 + \ldots + X_n) - b_n \xrightarrow{d} Z \tag{6}$$

*if and only if $Z$ is stable, in which case $a_n = n^{-1/\alpha}$ for some $0 < \alpha \leq 2$.*

A major drawback to use the stable distributions in practice is the fact that for all but a few stable distributions (Gaussian, Cauchy, Lévy) there is no closed form for the density or the distribution function. However, the stable distributions can be characterized by their characteristic function, see e.g. [6].

**Definition 2.3.** *A variable $X$ is a standard stable variable, or*

$$X \sim S_\alpha(1, \beta, 0) \tag{7}$$

*if its characteristic function equals*

$$\varphi(t) = \mathrm{E}\left[e^{itX}\right] = \exp\left\{-|t|^\alpha \, \omega_{\alpha,\beta}(t)\right\} \tag{8}$$

*where*

$$\omega_{\alpha,\beta}(t) = \begin{cases} 1 - i\beta \, \mathrm{sign}(t) \tan(\pi\alpha/2) & \text{if } \alpha \neq 1 \\ 1 + i\beta \frac{2}{\pi} \, \mathrm{sign}(t) \ln|t| & \text{if } \alpha = 1 \, . \end{cases} \tag{9}$$

**Definition 2.4.** *A variable $Y$ is a (general) stable variable, or*

$$Y \sim S_\alpha(\gamma, \beta, \delta) \tag{10}$$

*if we have the equality in distribution*

$$Y \stackrel{d}{=} \begin{cases} \delta + \gamma X & \text{if } \alpha \neq 1 \\ \delta + \gamma X + \tau\gamma\beta\frac{2}{\pi}\ln|\gamma| & \text{if } \alpha = 1 \end{cases} \tag{11}$$

*with $X$ a standard stable variable.*

So, a general stable distribution requires four parameters to describe: an index of stability or characterisitic exponent $\alpha \in (0, 2]$, a skewness parameter $\beta \in [-1, 1]$, a scale parameter $\gamma > 0$ and a location parameter $\delta \in \mathbb{R}$. Note that for $\alpha = 2$ the variable $X$ is N$(0, 2)$ distributed, and the variable $Y$ is normally distributed with mean $\delta\tau$ and standard deviation $\gamma\sqrt{2}$.

From now on we will assume that $\alpha \neq 1$ in order not to complicate the formulas. The case where $\alpha = 1$ can be described in an analogous way. To simplify the notation with respect to the time scale, we will write $S_\alpha(\gamma, \beta, \delta; \tau)$ for $S_\alpha(\gamma\tau^{1/\alpha}, \beta, \delta\tau)$.

In the present contribution, we assume that the increments follow a stable law $S_\alpha(\gamma, \beta, \delta; t_i - t_{i-1})$. This implies that (3) is changed into

$$Y(t_i) - Y(t_{i-1}) \stackrel{d}{=} \delta(t_i - t_{i-1}) + (t_i - t_{i-1})^{1/\alpha}\gamma Z_i \tag{12}$$

with $Z_i \sim S_\alpha(1, \beta, 0; 1)$ independent standard stable variables.

Since we work with stable processes, for the total rate of return we have

$$Y(t_j) \stackrel{d}{=} \delta t_j + t_j^{1/\alpha}\gamma X_j \tag{13}$$

with $X_j \sim S_\alpha(1, \beta, 0; 1)$ again a standard stable variable. Just as in the Wiener case, the variables $Y(t_1), \ldots, Y(t_n)$ are dependent. For a choice of $\alpha = 2$, the normal model emerges.

Next, we introduce a risk parameter $\Theta$. Conditionally on this risk parameter, the distribution of the increments is the one of a stable law. We consider the compounded rate of return:

$$Y(t_j)|\Theta = \theta \sim S_\alpha(\gamma, \beta, \delta; t_j\theta) \; ; \tag{14}$$

In case $\Theta$ has all its mass in one, i.e. $\text{Prob}[\Theta = 1] = 1$, $Y(t_j)|\Theta = \theta$ reduces to the ordinary stable law.

The next lemma illustrates the *stability property* of random variables with stable distribution as defined in (14), and at the same time proves the result in (13).

**Lemma 2.5.** *Let the variables $Y_1$ and $Y_2$ be defined as*

$$Y_1|\Theta = \theta \stackrel{d}{=} \delta\tau\theta + (\tau\theta)^{1/\alpha}\gamma X_1 \tag{15}$$

$$Y_2|\Theta = \theta \stackrel{d}{=} \delta(t-\tau)\theta + ((t-\tau)\theta)^{1/\alpha}\gamma X_2 \tag{16}$$

*with $0 \le \tau \le t$ and with $X_1$ and $X_2$ independent standard stable variables. Then, conditionally on $\Theta$, the sum $\tilde{Y} = Y_1 + Y_2$ in distribution equals*

$$\tilde{Y}|\Theta = \theta \stackrel{d}{=} \delta t\theta + (t\theta)^{1/\alpha}\gamma\tilde{X} \tag{17}$$

*with $\tilde{X}$ a new standard stable variable.*

*Proof.* Although this result is well known, we give a proof for the sake of completeness. Conditionally on $\Theta$, the characteristic function can be written as

$$\begin{aligned}
&\mathrm{E}\left[e^{ik\tilde{Y}}|\Theta = \theta\right] \\
&= \mathrm{E}\left[e^{ik\left\{\delta\tau\theta + (\tau\theta)^{1/\alpha}\gamma X_1 + \delta(t-\tau)\theta + ((t-\tau)\theta)^{1/\alpha}\gamma X_2\right\}}\right] \\
&= e^{ik\delta t\theta}\cdot \mathrm{E}\left[e^{ik(\tau\theta)^{1/\alpha}\gamma X_1}\right]\cdot \mathrm{E}\left[e^{ik((t-\tau)\theta)^{1/\alpha}\gamma X_2}\right].
\end{aligned} \tag{18}$$

Making use of (8) and (9) for both $X_1$ and $X_2$, we find

$$\begin{aligned}
&\mathrm{E}\left[e^{ik\tilde{Y}}|\Theta = \theta\right] \\
&= e^{ik\delta t\theta}\cdot \mathrm{E}\left[e^{-k^\alpha\tau\theta\gamma^\alpha\left(1 - i\beta\ \mathrm{sign}(k(\tau\theta)^{1/\alpha}\gamma)\tan(\pi\alpha/2)\right)}\right] \\
&\quad \cdot \mathrm{E}\left[e^{-k^\alpha(t-\tau)\theta\gamma^\alpha\left(1 - i\beta\ \mathrm{sign}(k((t-\tau)\theta)^{1/\alpha}\gamma)\tan(\pi\alpha/2)\right)}\right] \\
&= e^{ik\delta t\theta}\cdot \mathrm{E}\left[e^{-k^\alpha t\theta\gamma^\alpha\left(1 - i\beta\ \mathrm{sign}(k(t\theta)^{1/\alpha}\gamma)\tan(\pi\alpha/2)\right)}\right].
\end{aligned} \tag{19}$$

From this intermediate result, it is immediately clear that

$$\mathrm{E}\left[e^{ik\tilde{Y}|\Theta=\theta}\right] = \mathrm{E}\left[e^{ik\left\{\delta t\theta + (t\theta)^{1/\alpha}\gamma\tilde{X}\right\}}\right] \tag{20}$$

with $\tilde{X}$ a standard stable variable. □

# 3 Convex upper bounds

In many financial and actuarial applications, the distribution of the (stochastic) quantity under investigation is too difficult to obtain. In the present case for example, the stochastic variables $Y(t_j)$ in (2) are dependent, since they are constructed as successive partial sums of several independent variables.

In such cases, the method of convex upper bounds is extremely helpful. The idea consists of replacing the incalculable exact distribution by a simpler approximate distribution of a random variable which is more dangerous than the original one.

The most important result regarding this idea is summarized in the following theorem, a proof of which can be found in [1].

**Proposition 3.1.** *Consider a sum of functions of random variables*

$$V = \phi_1(X_1) + \phi_2(X_2) + \ldots + \phi_n(X_n), \tag{21}$$

*where the functions $\phi_i$ are real functions. Then the variable*

$$W = F^{-1}_{\phi_1(X_1)}(U) + F^{-1}_{\phi_2(X_2)}(U) + \ldots + F^{-1}_{\phi_n(X_n)}(U) \tag{22}$$

*with $U$ a standard uniformly random variable, defines an upper bound in convexity order, or*

$$V \leq_{cx} W. \tag{23}$$

In the previous result, the notation $F_{X_j}(x)$ is used for the distribution function of $X_j$, i.e.

$$F_{X_j}(x) = \mathrm{Prob}(X_j \leq x); \tag{24}$$

the inverse function is defined as ($p \in (0,1)$):

$$F_{X_j}^{-1}(p) = \inf\{x \in \mathbb{R} : F_{X_j}(x) \geq p\}. \tag{25}$$

Note that if the function $\phi$ is strictly increasing, then $F_{\phi(X)}^{-1}(p) = \phi(F_X^{-1}(p))$. If $\phi$ is strictly decreasing, then $F_{\phi(X)}^{-1}(p) = \phi(F_X^{-1}(1-p))$.

For the concept of convex ordering and its properties, see [3] and the references therein.

# 4 Results for cash-flows

We now return to the present value of a series of (positive and/or negative) payments

$$A = \sum_{j=1}^{n} c_j e^{-Y(t_j)}. \tag{26}$$

The variables $Y(t_j)$ ($j = 1, \ldots, n$), representing the stochastic continuous compounded rates of return over the periods $[0, t_j]$, can be written as

$$Y(t_j) = \sum_{i=1}^{j} (Y(t_i) - Y(t_{i-1})) \qquad (0 = t_0 < t_1 < \ldots < t_n = t) \tag{27}$$

with, conditionally on $\Theta = \theta$,

$$Y(t_i) - Y(t_{i-1}) \stackrel{d}{=} \delta(t_i - t_{i-1})\theta + ((t_i - t_{i-1})\theta)^{1/\alpha}\gamma Z_i \,; \tag{28}$$

the random variables $Z_i$ are independent standard stable variables with distribution $S_\alpha(1, \beta, 0; 1)$, and the risk parameter $\Theta$ is independent of the variables $Z_i$.

As mentioned before, it follows from the model that, conditionally on $\Theta = \theta$,

$$Y(t_j) \stackrel{d}{=} \delta t_j \theta + (t_j \theta)^{1/\alpha} \gamma X_j \tag{29}$$

where now the variables $X_j$ are dependent standard stable variables.

## 4.1 General results

**Proposition 4.1.** *Let $U$ be a random variable which is uniformly distributed on $[0,1]$. For the present value $A$ in (26), the variable*

$$A_{upp} = \sum_{j=1}^{n} c_j e^{-\delta t_j \Theta - (t_j\Theta)^{1/\alpha}\gamma\,\mathrm{sign}(c_j)F^{-1}(U;\alpha,\mathrm{sign}(c_j)\beta)} \tag{30}$$

*where $F(x;\alpha,\beta) = \mathrm{Prob}(X_j \leq x)$ denotes the distribution function of a standard stable variable, defines an upper bound in convexity order, or*

$$A \leq_{cx} A_{upp} \,. \tag{31}$$

*Proof.* This follows directly from proposition 3.1 and from the symmetry property $F^{-1}(1-U;\alpha,\beta) = -F^{-1}(U;\alpha,-\beta)$. □

Starting from this result for the boundary variable, we arrive at an expression for the stop-loss premiums.

**Proposition 4.2.** *The stop-loss premiums of the present value $A$ in (26) are bounded from above by*

$$\begin{aligned}\mathrm{E}\left[(A-k)_+\right] \;\leq\; & \int_0^{+\infty} dF_\Theta(\theta) \int_0^{u_\theta(k)} du \\ & \left(\sum_{j=1}^{n} c_j e^{-\delta t_j\theta-(t_j\theta)^{1/\alpha}\gamma\,\mathrm{sign}(c_j)F^{-1}(u;\alpha,\mathrm{sign}(c_j)\beta)} - k\right)\end{aligned} \tag{32}$$

*where for each value of $k$ and $\theta$ the value $x_\theta(k)$ is defined implicitly through the equation*

$$\sum_{j=1}^{n} c_j e^{-\delta t_j \theta - (t_j\theta)^{1/\alpha}\gamma\,\mathrm{sign}(c_j)F^{-1}(u_\theta(k);\alpha,\mathrm{sign}(c_j)\beta)} = k \,. \tag{33}$$

*The function $F_\Theta(\theta)$ denotes the distribution function of the risk parameter $\Theta$.*

*Proof.* Because of proposition 4.1, we know that

$$\mathrm{E}\left[(A-k)_+\right] \leq \mathrm{E}\left[(A_{upp}-k)_+\right] \tag{34}$$

with

$$\begin{aligned}
\mathrm{E}\left[(A_{upp}-k)_+\right] &= \int_0^{+\infty} dF_\Theta(\theta) \int_0^1 du \qquad (35)\\
&\left(\sum_{j=1}^{n} c_j e^{-\delta t_j\theta-(t_j\theta)^{1/\alpha}\gamma\,\mathrm{sign}(c_j)F^{-1}(u;\alpha,\mathrm{sign}(c_j)\beta)} - k\right)_+ .
\end{aligned}$$

The desired result follows by observing that the sum in (35) is a decreasing function of $u$ since the terms are all decreasing functions of $u$. □

Once the stop-loss premiums are found, the distribution function can be easily determined. Indeed, there is a well-known link between stop-loss premiums and distribution, stating that

$$\frac{d}{dk}\mathrm{E}\left[(A-k)_+\right] = F_A(k) - 1\,, \qquad (36)$$

where the notations are obvious.

**Proposition 4.3.** *The cumulative distribution for the quantity $A_{upp}$ mentioned in proposition 4.1 can be calculated as*

$$F_{upp}(k) = \mathrm{Prob}[A_{upp} \le k] = 1 - \int_0^{+\infty} u_\theta(k)dF_\Theta(\theta) \qquad (37)$$

*with $u_\theta(k)$ defined implicitly in* (33).

*Proof.* This follows immediately when applying (36) to (32). □

Note that if all $c_j > 0$, then

$$F_{upp}(k) = \mathrm{Prob}[A_{upp} \le k] = 1 - \int_0^{+\infty} F(x_\theta(k);\alpha,\beta)dF_\Theta(\theta) \qquad (38)$$

with $x_\theta(k)$ defined implicitly through

$$\sum_{j=1}^{n} c_j e^{-\delta t_j\theta - (t_j\theta)^{1/\alpha}\gamma x_\theta(k)} = k\,. \qquad (39)$$

## 4.2 Special cases & model modifications

After presenting the general results, we want to specify the results for three special cases for the distribution of the variable $\Theta$. We will use the same three cases for the numerical illustrations in the next section.

1. *The risk parameter $\Theta$ has all its mass in one, or* $\mathrm{Prob}[\Theta = 1] = 1$.
   The model degenerates to the ordinary and unconditional stable model. The distribution function of the upper bound can be written as
   $$F_{upp}^{(1)}(k) = 1 - u(k) \tag{40}$$
   with the values $u(k)$ defined implicitly through the equation
   $$\sum_{j=1}^{n} c_j e^{-\delta t_j - t_j^{1/\alpha}\gamma\,\mathrm{sign}(c_j)F^{-1}(u(k);\alpha,\mathrm{sign}(c_j)\beta)} = k\,. \tag{41}$$
   If $\alpha$ is chosen equal to 2, we recover the results as mentioned in [2].
2. *The risk parameter $\Theta$ is exponentially distributed with unit mean.*
   The model is said to follow a geometric stable law. The variable $Y(t)$ can be seen as the sum of a stochastic number of independent standard stable variables, where the total number of terms follows a geometric distribution (see [4]).
   Now the distribution function of the upper bound can be written as
   $$F_{upp}^{(2)}(k) = 1 - \int_0^{+\infty} e^{-\theta} u_\theta(k)\, d\theta \tag{42}$$
   with the values $u_\theta(k)$ defined in (33).
3. *The risk parameter $\Theta$ only appears in the volatility term.*
   In this case the model slightly differs, and the rate of return $Y(t_j)$ is (conditionally on $\Theta = \theta$) distributed as
   $$Y(t_j) \stackrel{d}{=} \delta t_j + (t_j\theta)^{1/\alpha}\gamma X_j\,. \tag{43}$$
   The distribution function of the upper bound then equals
   $$F_{upp}^{(3)}(k) = 1 - \int_0^{+\infty} dF_\Theta(\theta) v_\theta(k) \tag{44}$$
   with $v_\theta(k)$ defined implicitly through
   $$\sum_{j=1}^{n} c_j e^{-\delta t_j - (t_j\theta)^{1/\alpha}\gamma\,\mathrm{sign}(c_j)F^{-1}(v_\theta(k);\alpha,\mathrm{sign}(c_j)\beta)} = k\,. \tag{45}$$

# 5 Numerical illustration

In this section, we will present a few figures with graphs of the distribution functions of the upper bounds for the present value (26), as given in (40), (42) and (44).

The use of stable laws brings about a difficulty, which has to be found in the fact that we do not have a closed form for their distribution function. In order to solve this problem, we will make use of a recent numerical algorithm proposed by Nolan (see [6]).

In Figure 1 we plot the distribution function of $A_{upp}$, in case of a cash-flow $c_t = 10$, $t = 1, \ldots, 10$, and with $\mathrm{Prob}[\Theta = 1] = 1$. The parameters of the stable distribution are $\alpha = 1.8$ and $\beta = -0.05$, while $\delta$ and $\gamma$ equal 0.07 and 0.10 respectively. The distribution function appears to be rather close to the distribution function of $A$, which was obtained by Monte Carlo simulation. In order to compare the accuracy in the tails, we construct a QQ-plot of the corresponding distributions. Figure 2 confirms the heavy-tailedness of the upper bound and indicates that the right quantiles are slightly overestimated. For instance, the relative error of the 99% quantile is approximately 8%.

Replacing the distribution of the risk parameter $\Theta$ by the Exp(1) distribution yields Figure 3. In Figure 4 we turn to the modified model (43) with $\Theta \sim \chi_1^2$. Again, both upper bounds prove to be good approximations for the corresponding exact distributions.

In Figures 5 and 6, we use the same model as in Figure 1, but we change the cash-flow to $c_t = 1, \ldots, 10$ and $c_t = 10, \ldots, 1$ respectively. In case of an increasing cash-flow, the upper bound seems to approximate the exact distribution slightly better than in case of a decreasing cash-flow.

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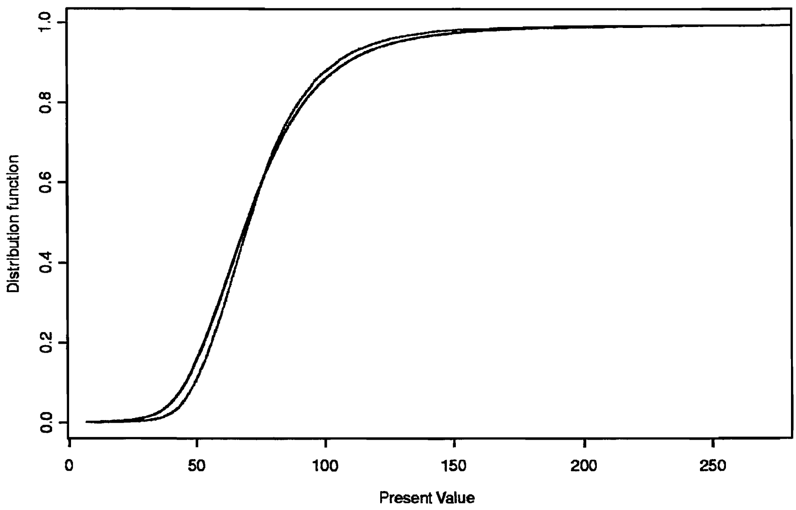


Figure 1: Distribution function of $A_{upp}$ (black) for $c_t = 10$ $(t = 1, \ldots, 10)$ and $\mathrm{Prob}[\Theta = 1] = 1$, compared to a simulated distribution function of $A$ (grey).

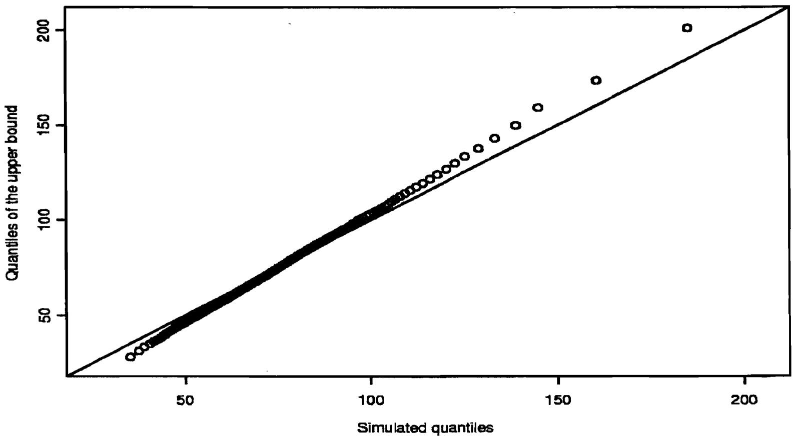


Figure 2: QQ-plot of $A_{upp}$ versus $A$, for $c_t = 10$ $(t = 1, \ldots, 10)$ and $\mathrm{Prob}[\Theta = 1] = 1$.

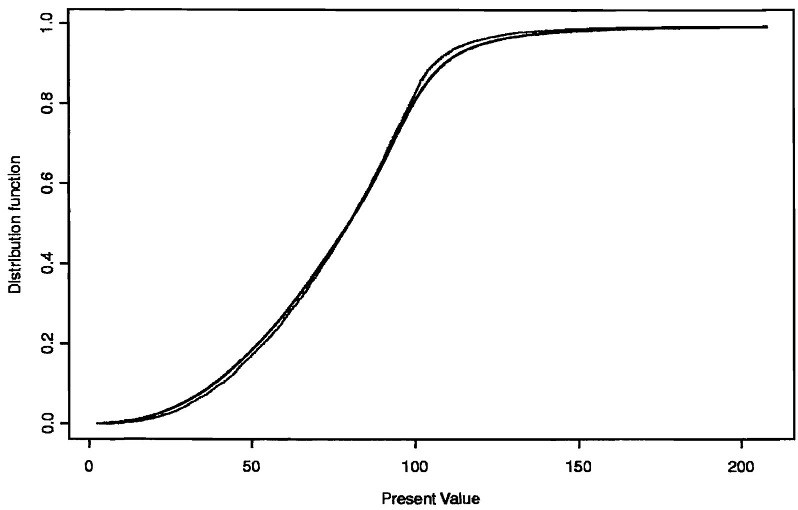


Figure 3: Distribution function of $A_{upp}$ (black) for $c_t = 10$ ($t = 1, \ldots, 10$) and $\Theta \sim$ Exp(1), compared to a simulated distribution function of $A$ (grey).

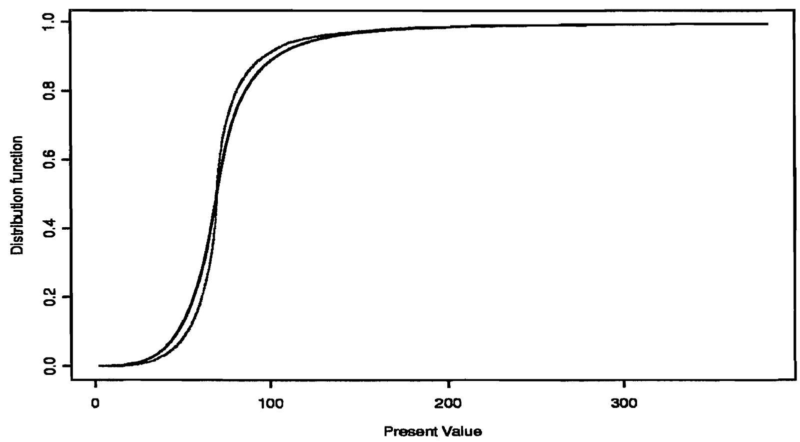


Figure 4: Distribution function of $A_{upp}$ (black) for $c_t = 10$ ($t = 1, \ldots, 10$) in special case 3 with $\Theta \sim \chi^2_1$, compared to a simulated distribution function of $A$ (grey).

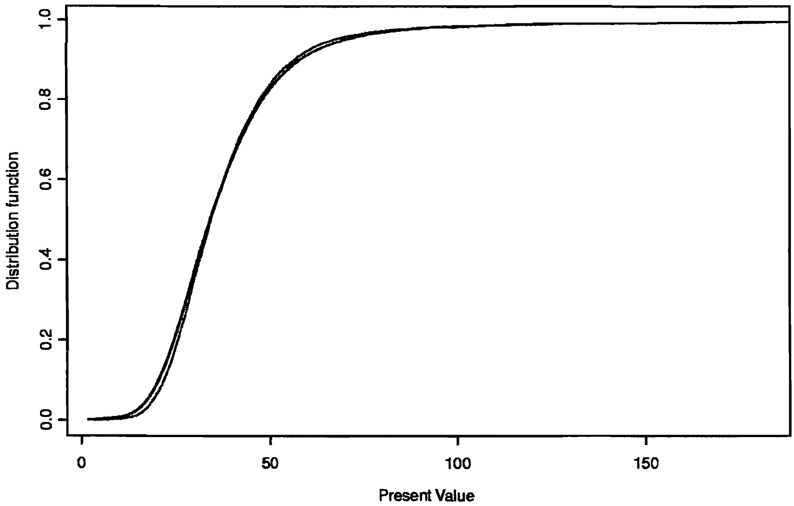


Figure 5: Distribution function of $A_{upp}$ (black) for $c_t = 1, \dots, 10$ and Prob$[\Theta = 1] =$ 1, compared to a simulated distribution function of $A$ (grey).

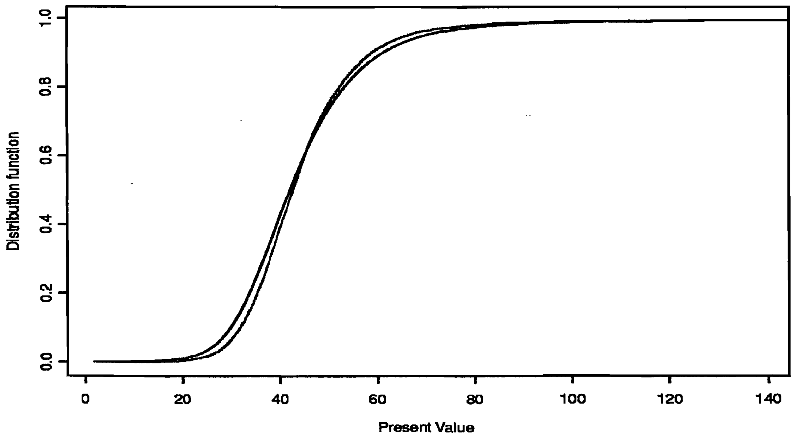


Figure 6: Distribution function of $A_{upp}$ (black) for $c_t = 10, \dots, 1$ and Prob$[\Theta = 1] =$ 1, compared to a simulated distribution function of $A$ (grey).